Optimal, Truthful, and Private Securities Lending

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Motivation

Motivated by challenges associated with **securities lending**, the mechanism underlying short selling of stocks in financial markets

We consider allocation of a **scarce commodity** in settings in which **privacy concerns** or **demand uncertainty** may be in conflict with **truthful reporting**

Goal is to construct a privacy protecting allocation mechanism that motivates truthful reporting without sacrificing too much utility

Contributions

1. Without privacy constraints, we construct an optimal greedy allocation rule for which truthfulness is a dominant strategy
2. In order to guarantee clients an appropriate notion of privacy, we reformulate the allocation rule as an ascending price auction in which clients cannot collude to infer too much information about any one bidder
3. Finally, we expand this into an allocation mechanism that can handle arbitrary and adaptive client request strategies while still providing privacy and near optimality and incentivizing truthfulness

Setting

- Lender distributes up to V shares to n clients over time horizon T at fixed price per unit
- Each client i has **non-strategic** distribution over usages, U_{it}
- Client has **strategic** distribution over requests, $Q_{it}(r_{it}|u_{it})$
- At each time t, client i draws $u_{it}, r_{it} \sim Q_{it}(u_{it}, r_{it})$, but **only request, r_{it}, is visible to lender**
  - We consider a distribution for client i at time t **truthful** if $Q_{it}(r_{it}|u_{it}) = 1$ if $u_{it} = r_{it}$ and $Q_{it}(r_{it}|u_{it}) = 0$ otherwise
  - Below are two strategic choices of $Q_{it}(r_{it}|u_{it})$ for client i

Table: Sample Truthful Distribution

u_{it} \ r_{it}	0	1	2
0	$\frac{1}{3}$	0	0
1	0	$\frac{1}{3}$	0
2	0	0	$\frac{1}{3}$

Table: Sample Untruthful Distribution

u_{it} \ r_{it}	0	1	2
0	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
1	0	$\frac{1}{6}$	$\frac{1}{6}$
2	0	0	$\frac{1}{3}$

Clients' and Lender's Goals

Definition

An allocation rule A is a one-shot algorithm that maps a set of requests (r_{it}) and conditional distributions $Q_{it}(\cdot|u_{it})$ on r_{it} to an allocation S_t

Lender's Goal: Choose *allocation rule* A to maximize lender's utility

- Lender's utility for allocation rule A is expected number of shares used
- Allocation rule assumes full knowledge of conditional distributions $Q_i(r_i|u_i)$, which could be estimated from a client's history
- Given knowledge of Q_i, lender can compute the posterior distribution $Q_i(u_i|r_i)$ on the true demand u_i given r_i, via Bayes' rule:

Client's payoff is number of shares actually used: if client i is allocated s_{it} shares in an allocation S_t, the payoff is

$$v_i(S_t) = min(s_{it}, u_{it})$$

Greedy Allocation Rule

Algorithm maximizing lender's utility (Greedy): *Greedy* operates by sequentially assigning shares $1 \ldots V$, where each share is assigned to the client i most likely to utilize one additional share (based on $Q_i(u_i|r_i)$)

Results: The allocation returned by *Greedy*, S, maximizes the expected payoff for the lender. Given that the lender is solving the allocation problem optimally for the reported Q distributions, *truth telling is a dominant strategy*

Theorem: Fix a set of choices Q_{-i} and reports r_{-i} for all clients other than i, and a realization of client i's usage $u_i \sim U_i$. Let Q_i^T denote the truthful strategy $Q_i^T(r_i|u_i) = \mathbf{1}_{r_i}$, and let $Q_i(r_i|u_i)$ denote any other strategy. Let A denote the lender's optimal allocation. Then:

$$v_A^i(Q_i) \leq v_A^i(Q_i^T)$$

Auction Formulation

- We also seek to understand situations in which clients have **privacy concerns** and possibly an **adaptive request strategy**
- First, we re-conceptualize the problem of computing the optimal allocation for the lender given known distributions Q_i as computing the **social welfare maximizing allocation** with respect to a set of valuation functions for each client i
- We then give an algorithm that uses an **ascending price auction formulation** to compute an **approximately optimal allocation**, which can be adapted to satisfy **(joint) differential privacy**

Auction Setting

- Consider a more general setting in which V identical units of a good are being sold to n bidders
- Each bidder has arbitrary decreasing marginal valuation function for up to U units of each good
- **Goal**: Wish to find welfare maximizing allocation

Auction Rule

- Ascending price auction (*Auction*) works by sequentially allowing bidders to claim an additional unit of the good if the current price is below their specified marginal utility for that unit
- Price increments by α after every V bids
- *Auction* terminates when there are no more bids

Auction Rule Guarantee

Theorem

Auction(V, α, U) terminates after at most $\frac{V}{\alpha} + 1$ rounds. At termination, S constitutes an $\frac{\alpha V}{n}$-optimal allocation:

$$v(S) \geq \max_{S'} v(S') - \frac{\alpha V}{n}$$

- Still no notion of privacy
- What we want here is **joint differential privacy**, which in an intuitive sense, prevents adversaries from learning too much about agent i by observing the allocation to agents other than i (or from all other agents colluding)

Private Auction Formulation

We modify *Auction* as follows to make it jointly differentially private:

1. Running count of total number of bids placed so far is computed approximately using a differentially private estimator
2. Auction stops early, based on some accuracy parameter fed to algorithm
3. Run the auction with a supply of $V - E$ shares, where E corresponds to the maximum error of our differentially private bid counter (ensures computed allocation is always feasible)

Private Auction Guarantees

Private Auction obtains the following results:

- For sufficiently large auctions, we can **achieve privacy** while still outputting a **high-quality allocation** (near optimal welfare).
- Private auction remains **approximately dominant-strategy truthful**.

Allocation Mechanism

Finally, we form an approximately optimal and approximately private allocation mechanism that can handle **adaptive strategies** by clients

Definition

An allocation mechanism $\mathcal{A}$ maps the requests $r_t = (r_{it})$ at time t and the history H_t to allocations of shares: $\mathcal{A}(r_{1t}, \ldots, r_{nt}; H_t) = S_t$.

- Now, each client i has the freedom to (adaptively) choose an arbitrary mapping $L_i^t : \mathcal{H}_t^i \times [U] \to [R]$ that maps the realized history and demand H_t, u_{it} respectively, to a request r_{it}.
- The utility of client i is defined as:

$$v_{\mathcal{A}}^i(L_1^i, \ldots, L_T^i) = \sum_{t=1}^{T} \mathbb{E}[v_i(\mathcal{A}(r_{it}, r_{-it}; H_t)],$$

- Lender's utility now incorporates clients' histories:

$$v(A) = \sum_t \sum_i \mathbb{E}_{u_{it} \sim Q_{it}(u_{it}|r_{it}, H_t), A} \min(A(r_1, \ldots, r_n; Q_1, \ldots, Q_n)_i, u_{it})$$

Private Allocation Mechanism

Algorithm 1 Greedy Private Mechanism

procedure $\mathcal{A}$(Utility distributions $U_i \in \Delta([U])$ for n clients, V shares to allocate at each of T rounds, $\textsc{PrivAuc}, \epsilon, \alpha$)
  for $t = 1 \ldots T$ **do**
    for i = 1 ... n **do**
      Client i draws $u_{it} \sim U_i$
      Client i picks request distribution $Q_{it} = L_t^i(\mathcal{H}_t^i, u_{it})$
      Client i draws $r_{it}|u_{it} \sim Q_{it}(r_{it}|u_{it})$, and submits r_{it}
    $\mathcal{A}$ updates its estimates $\hat{Q}_i(r_{it}) = \mathbf{1}_{r_{it}}$
    $\mathcal{A}$ computes allocation $S_t = \textsc{PrivAuc}(\hat{Q}_1(r_{1t}), \ldots \hat{Q}_t(r_{nt}), \epsilon, \alpha)$
    $\mathcal{A}$ observes the executed shares $v_i(S_t)$ for each client
    $\mathcal{A}$ updates its estimates of the conditionals $\hat{Q}_i(r_{it})$
    $\mathcal{A}$ updates the history: $H_{t+1} = H_t \cup (r_{it}, s_{it}, v_i(S_t))_{i=1}^n$

Summary

Theorem: Let A be a private auction with appropriate values of U, V, ϵ and ρ such that A is $(\epsilon', \beta/T)$-JDP with $\epsilon' = \tilde{O}(\epsilon/\sqrt{T})$ and outputs S such that $E[V(S)] \geq (1-\rho)OPT_V - \rho$. Take β, ρ such that $\sqrt{\beta + (1-\beta)\rho} \leq \beta^2/T$. Then for a $(1-\beta)$ fraction of the n clients i, let L^t_{i*} denote the truthful strategies, and let L^t_i be any other set of strategies. Then a private greedy allocation rule for the private auction satisfies:

$$v_i(L^1_i, \ldots, L^n_i) \leq e^{2\epsilon} v_i(L^1_{i*}, \ldots, L^n_{i*}) + 2\beta UT + e^{\epsilon}\frac{\beta^2}{1-\beta^2/T}$$

$$v_A(L^t_{i*}) \geq (1-\rho)OPT_V - \rho T,$$

where OPT_V denotes the lender's optimal utility.

Joint differential privacy in our allocation mechanism enforces **truthfulness** as an approximately dominant strategy and guarantees **near optimality**

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